WHAT RESEARCH TELLS US ABOUT TEACHING MATHEMATICS THROUGH PROBLEM SOLVING

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To Appear: In F. Lester (Ed.), Research and issues in teaching mathematics through problem solving. Reston, VA: National Council of Teachers of Mathematics.

The author thanks Vic Cifarelli, Jim Hiebert, Frank Lester, and Judi Zawojewski for helpful discussions in the process of preparing this chapter. Preparation of this chapter is partially supported by a grant from the National Science Foundation (ESI-0114768). However, any opinions expressed herein are those of the author and do not necessarily represent the views of NSF.

WHAT RESEARCH TELLS US ABOUT TEACHING MATHEMATICS THROUGH PROBLEM SOLVING

The teaching of problem solving has a long history in school mathematics (D'Ambrosio & Lester, this volume; Stanic & Kilpatrick, 1988). In the past several decades, there have been significant advances in the understanding of the complex processes involved in problem solving (Lester, 1994; Schoenfeld, 1992; Silver, 1985). There also has been considerable discussion about teaching mathematics with a focus on problem solving (e.g., Hembree & Marsh, 1993; Henningson & Stein, 1997; Hiebert et al., 1997; Kroll & Miller, 1993; Stein, Smith, & Silver, 1999). However, teaching mathematics *through* problem solving is a relatively new idea in the history of problem solving in the mathematics curriculum (Lester, 1994). In fact, because teaching mathematics through problem solving is a rather new conception, it has not been the subject of much research. Although less is known about the actual mechanisms students use to learn and make sense of mathematics through problem solving, there is widespread agreement that teaching through problem solving holds the promise of fostering student learning (Schroeder & Lester, 1989). Many of the ideas typically associated with this approach (e.g., changing the teacher's roles, designing and selecting problems for instruction, collaborative learning, problematizing the curriculum) have been studied extensively, and research-based answers to various frequently asked questions about problem-solving instruction are now available.

Issues and Concerns Related to Teaching Through Problem Solving

This chapter discusses four issues and concerns related to teaching through problem solving. These four issues are related to four commonly asked questions: (1) Are young children really able to explore problems on their own and arrive at sensible solutions? (2) How can teachers learn to teach through problem solving? (3) What are students' beliefs about teaching through problem solving? (4) Will students sacrifice basic skills if they are taught mathematics through problem solving? In the discussion of each issue, available research evidence that addresses the issue is first reviewed, and then research needed to address the issue more completely is suggested.

Issue 1: Are Young Children Really Able to Explore Problems on Their Own and Arrive at Sensible Solutions?

Teaching through problem solving starts with a problem. Students learn and understand important aspects of the concept or idea by exploring the problem situation. The problems used tend to be more open-ended and allow for multiple correct answers and multiple solution approaches. In teaching through problem solving, problems not only form the organizational focus and stimulus for students' learning, but they also serve as a vehicle for mathematical exploration. Students play a very active role in their learning—exploring problem situations with teacher guidance and "inventing" their own solution strategies. In fact, the students' own exploration of the problem is an essential component in teaching through problem solving. For example, in curriculum projects designed to help students in primary grades learn and understand number concepts and operations with understanding, the learning of number concepts and operations is perceived as a "conceptual problem-solving activity" in which teachers support students' efforts to work out their own procedures and rules related to addition and subtraction (Fuson et al., 1997). However, a fundamental question arises: Are students really capable of exploring problem situations and inventing strategies to solve the problems?

Many researchers (e.g., Carpenter, Franke, Jacobs, Fennema, &Empson, 1998; Kamii, 1989; Maher & Martino, 1996; Resnick, 1989) have investigated students' mathematical thinking and indicated that young children can explore problem situations and "invent" ways to solve the problems. For example, traditionally, to find the sum 38 + 26, students are expected to add the ones (8 + 6 = 14), and write down 4 for the unit place of the sum and carry over 1 to the ten's place. Carpenter et al. (1998) found that many first-, second-, and third-grade students were able to use the following invented strategies to solve the problem: (1) "Thirty and twenty is fifty and the eight makes fifty eight. Then six more is sixty-four"; (2) "Thirty and twenty is fifty, and eight and six is fourteen. The ten from the fourteen makes sixty, so it is sixty four"; (3) "Thirty-eight plus twenty-six is like forty and twenty-four, which is sixty-four." In their study, Carpenter et al. (1998) found that 65% of the students in their sample had used an invented strategy before standard algorithms were taught. By the end of their study, 88% of their sample had used invented strategies at some point during their first three years of school. They also found that students who used invented strategies before they learned standard algorithms demonstrated better knowledge of base-ten number concepts and were more successful in extending their knowledge to new situations than were students who initially learned standard algorithms.

Recently, some researchers (e.g., Ben-Chaim et al., 1998; Cai, 2000) have also found evidence that middle school students are able to use invented strategies to solve problems. For example, when U.S. and Chinese sixth-grade students were asked to determine if each girl or each boy gets more pizza when seven girls share two pizzas and three boys share one pizza equally, they used eight different correct ways to justify that each boy gets more than each girl (Cai, 2000).

Collectively, the aforementioned studies not only demonstrate that students are capable of inventing their own strategies to solve problems, but they also show that it is possible to use the students' invented strategies to enhance their understanding of mathematics. Thus, it seems clear that students in elementary and middle schools are capable of inventing their own strategies to solve problems. However, there are at least two unanswered questions.

<u>Unanswered questions related to issue 1</u>. The first question has to do with students' invented strategies. In classrooms using problem-based inquiry (e.g., Carpenter et al., 1998; Cobb et al., 1991), students are given opportunities to use and discuss alternative strategies for solving problems before being taught any specific strategies. The question is: How do students learn to use

invented strategies in the first place before any instruction takes place? What kinds of experiences and knowledge do students draw upon to create sensible strategies? Kamii (1989) has argued that "the procedures children invent are rooted in the depth of their intuition and their natural ways of thinking" (p. 14). Clearly, we need to learn much more about what students' "natural" ways of thinking in mathematics are. We also need to determine if these natural ways are content or grade-level dependent.

The second unanswered question has to do with the efficiency of students' invented strategies: When students develop inefficient strategies, how can they be helped to develop more efficient strategies? Previous research has shown that students are capable of inventing problem-solving strategies or mathematical procedures, but the research also has shown that invented strategies are not necessarily efficient strategies (Cai, Moyer, & Grochowski, 1999; Carpenter et al., 1998; Resnick, 1989). For example, in a study by Cai et al. (1999), a group of middle school students was asked to solve the following problem involving arithmetic average. One student came up with an unusual strategy to solve it. In this solution, the student viewed throwing away the top and bottom scores as taking 15 away from each of the other scores. By inventing this approach, this student demonstrated an incredible understanding of averaging. However, this approach is somewhat inefficient. Clearly, invented strategies can serve as a basis for students' understanding of mathematical ideas and procedures but, based on their level of understanding, students also should be guided to develop efficient strategies.

Problem*:*
The average of Ed's ten test scores is 87. The teacher throws out the top and bottom scores, which are 55 and 95. What is the average of the remaining set of scores?

Description of the Solution:
The student first used one of the properties of average and determined that the average for the remaining eight scores must be between 55 and 95. Then the student drew a row of ten circles and put 95 in the first and 55 in the last, leaving eight empty circles. Using a modified sharing approach, the student realized that 55 and 95 contributed 15 to the average [(95 + 55) ÷ 10 = 15]. So the student said that each of the eight blank spaces should get 15. But since15 is 72

less than 87 (the average for the ten scores), the student then multiplied 10 by 72 and got 720 and then divided 720 by 8 to get 90. Thus, 90 became the average of the remaining eight scores after the top and bottom scores were thrown away.

Issue 2: How Can Teachers Learn to Teach through Problem Solving?

In their chapter in this volume, Curcio and Artzt discuss the various roles teachers play in teaching through problem solving. Unfortunately, little research is available about how teachers learn these roles. However, research does inform us in some important ways. It indicates that teachers' success in teaching through problem solving is related to the encouragement and support they receive from their fellow teachers and other resource partners as they begin to change their approach to teaching (Cohen, 1990; Cobb et al., 1991; Greeno & Goldman, 1998; Ma, 1999; Stein et al., 1999; Stigler & Hiebert, 1999). Teachers learn their new roles through teaching and self-reflection, rather than merely taking courses (Ball, 1993; Borko & Putman, 1996; Bransford, Brown, & Cocking, 2000; Shimahara & Sakai, 1995). After a series of studies of elementary and secondary preservice teachers, Ball (1993) concluded that requiring teachers to take more courses would not improve their understanding of school mathematics and enhance their teaching. Instead, teachers need opportunities to analyze mathematical ideas and make connections in instructional situations. Also, teachers learn to play their roles in teaching through participating in daily collegial activities in school (Paine & Ma, 1993; Stein et al., 1999). Shimahara and Sakai (1995) found that both U.S. and Japanese new teachers learn more about teaching from daily conversations with other teachers than from formally organized workshops or from student teaching. Everyday conversation about teaching in the school setting is important since the conversation is concrete and contextual rather than abstract and context free.

Two new roles that teachers are asked to play in a classroom based on teaching through problem solving involve selecting appropriate tasks and organizing classroom discourse. Research suggests the importance of selecting appropriate tasks and organizing classroom discourse for fostering students' mathematical understanding. Research also offers some insights about the factors that influence on teachers' selection of appropriate tasks and organization of classroom discourse in teaching through problem solving.

Selecting appropriate problems. As Van de Walle points out in his chapter in this volume, in teaching through problem solving, students actively participate in the process of knowledge construction and, therefore, make sense of mathematics in their own terms. In other words, they become active participants in the creation of knowledge rather than passive receivers of rules and procedures. While teaching through problem solving starts with problems, only worthwhile problems give students the chance both to solidify and extend what they know and to stimulate their learning. Therefore, one of the teachers' roles is to select or develop such worthwhile problems.

Doyle (1988) has argued that problems with different cognitive demands are likely to induce different kinds of learning. Problems govern not only students' attention to particular aspects of content, but also their ways of processing information. Regardless of the context, worthwhile tasks should be intriguing, with a level of challenge that invites exploration, speculation, and hard work (National Council of Teachers of Mathematics [NCTM], 2000, p. 19). Mathematical problems that are truly problematic and involve significant mathematics have the potential to provide the intellectual contexts for students' mathematical development.

But, what is a worthwhile problem? Lappan and Phillips (1998) have developed a set of criteria useful for choosing problems for middle grades' mathematics instruction. These criteria can also be applied to the selection of problems for instruction in earlier grades. It goes without saying that the most important criterion of a worthwhile mathematical problem is that the problem should serve as a means for students to learn important mathematics. Such a problem does not have to be complicated with a fancy format. As long as a problem can reach the goal of fostering students' learning important mathematics, it is a worthwhile problem. As Hiebert et al. (1996) have noted that a problem as simple as finding the difference in heights between two children, one 62 and the other 37 inches tall, can be a worthwhile problem if teachers use it appropriately for students' learning of multi-digit addition.

Organizing classroom discourse. Using worthwhile problems is an important, but not sufficient, feature of effective problem-based mathematics instruction because a worthwhile problem may not be implemented appropriately. For example, Stein, Grover, and Henningsen (1996) found that only about 50% of the tasks that were set up to require the application of procedures with meaningful connections were implemented in a way that resulted in meaningful connections. Therefore, another role teachers play is to decide how to use worthwhile problems to maximize students' learning opportunities in the classroom. In addition to engaging students in good problem-solving activities, the type of engagement is vitally important. Put another way, the nature of the classroom discourse--involving both students and teacher--is a very important consideration. In learning through problem solving, students not only have more opportunity to express their ideas and justify their answers verbally, but they also have more opportunities to engage in cognitively demanding questions (Hiebert & Wearne, 1993; Lampert, 1990).

There are a number of factors that may influence the implementation of worthwhile problems in the classroom. One of the predominant factors is the amount of time allocated to solving problems and discussing solution efforts (Henningsen & Stein, 1997, Perry, Vanderstoep, & Yu, 1993; Stigler & Hiebert, 1999). In teaching through problem solving, the discussion of a problem and its alternative solutions usually takes longer than the demonstration of a routine classroom activity. Hiebert and Wearne (1993) found that classrooms with a primary focus on teaching through problem solving used fewer problems and spent more time on each of them, compared to those classrooms without a primary focus on problem solving. Moreover, in these problem-solving classrooms, teachers ask more conceptually-oriented questions (e.g., describe a strategy or explain underlying reasoning for getting an answer) and fewer recall questions than teachers in the classrooms without a primary focus on problem solving. The findings are consistent with what has been found in cross-cultural comparative studies (Perry, Vanderstoep, & Yu, 1993; Stigler & Hiebert, 1999).

In teaching through problem solving, the teachers' role in guiding mathematical discourse is a highly complex activity. Besides devoting an appropriate amount of time to the discussion of problems, "teachers must also decide what aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them and thus eliminating the challenge" (NCTM, 2000, p. 19). In other words, it is important for teachers to provide enough support for students' mathematical exploration, but not so much support that they take over the process of thinking for their students (e.g., Ball, 1993; Lampert, 1985, Hiebert et al., 1997). There are no specific, research-based guidelines that teachers can use to achieve the appropriate balance between teacher-directed and teacher-guided instruction and it is unlikely that research will ever be able to provide such guidelines.

Not only do teachers need specific ideas about how they learn to play their roles, but they also need concrete examples to guide their practice. We need to document more cases and describe what teaching through problem solving looks like, how appropriate mathematical problems are selected, and how classroom discourse is organized to appropriately guide students engaging in the mathematical problem (Ball & Bass, 2000).

Issue 3: What Are Students' Beliefs about Problem Solving?

All too often students hold the belief that there is only one "right" way to approach and solve a problem. The results from both national (Lindquist, 1989) and international (Lapointe, Mead, &

Askew, 1992) assessments show that many students do not view mathematics as a creative and intellectually engaging activity, but rather as a set of rules and procedures that they must memorize in order to follow the single correct way rapidly to obtain the single correct answer. For example, nearly 50% of U.S. students reported that learning mathematics is mostly memorizing and about one-fifth of students disagreed with the statement that a mathematical problem can be solved in different ways (Lindquist, 1989).

Students' beliefs about the nature of problem solving are not restricted to how problems are supposed to be solved. Many students also have firmly held beliefs about what is expected of them when their teachers give them problems to solve. For example, in solving an absurd problem like "There are 26 sheep and 10 goats in a ship. How old is the captain?" Ten percent of Belgian kindergartners and first graders "solved" this problem by adding the numbers to get the captain's age (Verschaffel & De Corte, 1997). The percentages of students who "solved" the problem in this way increased to 60% for the third and fourth graders and 45% for the Belgian fifth graders. The more formal education the third, fourth, and fifth grade students had, the less attention they paid to making sense of the problem and their solutions, in contrast to the first graders. A similar problem was administered to a group of Chinese fourth graders, seventh graders, eighth graders, and twelfth graders. About 90% of the Chinese fourth graders, 82% of the seventh and eighth graders, and 34% of the twelfth graders "solved" this problem by combining numbers in it without realizing the absurd nature of the problem (Lee, Zhang, and Zheng, 1997). When these Chinese students were asked why they did not recognize that the problem was meaningless, many students responded that "any problem assigned by a teacher always has a solution." This sort of result has been documented consistently by other researchers (e.g., Lester, Garofalo, & Kroll, 1989).

Students' beliefs about problem solving can also be revealed when they are asked to solve a problem using alternative strategies. Some students seemed unconcerned about getting different answers for a problem with a unique answer when they are asked to solve the problem using different strategies. For example, in a study by Silver, Leung, and Cai (1995), Japanese and U.S. fourth-grade students were asked to find multiple ways to determine the total number of marbles that had been arranged in a certain way. Some students obtained different numerical answers when they used alternative solution strategies; surprisingly, they seemed unconcerned about getting different answers.

On the other hand, studies by a number of researchers (e.g., Carpenter et al., 1998; Cobb et al., 1991; Verschaffel & De Corte, 1997) suggest that it is possible to change students' beliefs about mathematics and problem solving using alternative instructional practices, such as teaching through problem solving. For example, in contrast to students in a control group, Cobb et al. (1991) found that students in their problem-centered project held more positive beliefs about the importance of understanding. Therefore, it is quite possible that teaching through problem solving may provide the kind of healthy learning environment for students to form positive beliefs about mathematics and problem solving before they develop any negative dispositions.

Unanswered questions related to issue 3. It is well-documented that teachers' beliefs about mathematics impact their teaching (Thompson, 1992). Teachers who hold different beliefs about mathematics teach differently. Then, what about students? How do students' beliefs about problem solving impact their learning in teaching through problem solving? Research clearly shows that some students do not believe that a mathematical problem can be solved in different ways, and they think that learning mathematics is mostly memorizing. How would students who hold such negative beliefs about problem solving learn mathematics differently from those who have more positive beliefs?

Another unanswered question is related to students' internal struggles when they are shifted from traditional to problem-based classroom. Research shows that when teachers shift from their usual ways of teaching to problem-based teaching, they face a number of dilemmas and internal struggles to make the change (Smith, 1995). If students have already become used to one way of instruction and suddenly start to experience a very different form of instruction in which they are expected to explore mathematics rather than follow procedures, what dilemmas do they face?

Issue 4: Will Students Sacrifice Basic Skills if They Are Taught Mathematics through Problem Solving?

In teaching through problem solving, students have opportunities to explore problem situations and solve problems, and they are encouraged to use whatever solution strategies they wish. Students also are given opportunities to share their various strategies with each other. Thus, students' learning and understanding of mathematics can be enhanced through considering one another's ideas and debating the validity of alternative approaches. In teaching through problem solving, the focus is on conceptual understanding, rather than on procedural knowledge; it is expected that students will learn algorithms and master basic skills as they engage in explorations of worthwhile problems. However many people, parents and teachers alike, worry that the development of students' higher-order thinking skills in teaching through problem solving comes at the expense of the development of basic mathematical skills. Obviously, both basic skills and high-order thinking skills in mathematics are important, but having basic mathematical skills does not imply having higher-order thinking skills or vice versa. Therefore, it is reasonable to question if students will sacrifice basic skills in a learning environment involving teaching through problem solving (Battista, 1999; Schoenfeld, 2002).

Several studies involving elementary students (Carpenter et al., 1998; Cobb et al., 1991; Fuson et al., 2000; Hiebert & Wearne, 1993) have consistently shown that in comparison to students in control groups, students experiencing problem-based instruction usually have higher levels of mathematical understanding and problem-solving skills and have at least comparable basic numerical skills. For example, Fuson et al. (2000) found that students using a problem-based curriculum outperformed students using a traditional curriculum on a computation test. On a number-sense test, students using the problem-based curriculum scored higher than traditional students on two tasks. In another study, Cobb et al. (1991) examined the performance on a standardized mathematics achievement test of ten classes, whose students had participated in a year-long, problem-centered mathematics project and compared them with eight non-project classes. They also studied the performance of these same classes of students on instruments designed to assess students' computational proficiency and conceptual development in arithmetic. They found that levels of computational performance between project and non-project students were comparable, but the project students had higher levels of conceptual understanding in mathematics than did non-project students. In a subsequent longitudinal study, Wood and Sellers (1997) analyzed the mathematical performance of three groups of elementary school students: those students who had received two years of problem-centered mathematics instruction, those who had received one year, and those who had received only traditional textbook instruction. The results of the analyses indicate that after two years in problem-centered classes, students had significantly higher achievement on standardized achievement measures and better conceptual understanding than did those students who had received traditional instruction. Other studies involving elementary school students (e.g., Carpenter et al., 1998; Hiebert & Wearne, 1993) have obtained similar results: Students learning mathematics through problem solving do at least as well as those students receiving traditional instruction on both basic computation and conceptual understanding.

Similarly, the few existing studies involving middle school students (Ridgeway, Zawojewski, Hoover, & Lambdin, 2002; Romberg & Shafer, 2002) have shown that students with problem-based instruction have higher levels of mathematical understanding than students with more traditional instruction, and there are comparable basic number skills between the two groups. For example, results from the study by Romberg and Shafer (2002) suggest that students using problem-centered curricular materials can maintain basic number skills while developing higher-order thinking skills. Ridgeway et al. (2002) obtained similar results when they compared the performance of middle school students who had received instruction following principles of teaching through problem solving with students who used more traditional middle school mathematics curricula. Specifically, they compared the two groups' performance on two tests: Iowa Test of Basic Skills (ITBS) to assess basic skills and a test designed by the Balanced Assessment Project to assess students' performance in mathematical reasoning, communication, connections, and problem solving. They found that the problem-based students showed significantly more growth than did students receiving traditional instruction on mathematical reasoning, communication, making connections, and problem solving while the growth of basic skills of the two groups was the same as the ITBS national sample.

Unanswered questions related to issue 4. Research involving elementary and middle school students seems to indicate that students will not sacrifice basic skills if they are taught mathematics through problem solving[1]. However, it is still early to make hard-and-fast claims with absolute confidence about the computational proficiency of students in a learning environment that focuses on teaching through problem solving. There are a number of unanswered questions that should be explored. First, How can we develop students' basic mathematical skills in teaching through problem solving? Research makes clear that meaningless rote memorization is not an

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[1] Results from research involving high school students are not consistent. For example, in a comparative study of the effects of the high school Core-Plus Mathematics Project (CPMP) curriculum and more conventional curricula on growth of student understanding, skill, and problem-solving ability in algebra, Huntley, Rasmussen, Villarubi, Sangtong, and Fey (2000) found that the CPMP curriculum is more effective than conventional curricula in developing student ability to solve algebraic problems when those problems are presented in realistic contexts and when students are allowed to use graphing calculators. Conventional curricula are more effective than the CPMP curriculum in developing student skills in manipulation of symbolic expressions in algebra when those expressions are presented free of application context and when students are not allowed to use graphing calculators.

appropriate way for students to learn basic facts and develop computational proficiency (Hiebert, 1999). The development of students' basic mathematical skills when teaching through problem solving is followed appears to be both sound and viable, but adequate descriptions of such teaching and its impact should be studied and documented.

Another unanswered question is related to the expectation of students' level of proficiency with basic mathematical skills. Research has shown that elementary and middle school students who have learned mathematics with problem-based instruction outperform their counterparts in traditional programs on tasks assessing higher-order thinking skills, without sacrificing basic mathematical skills. Apparently, however, the growth in higher-order thinking skills is greater than that in the basic skills. Should we be satisfied with this level of proficiency? Although this question is decided by value judgments rather than by merely empirical research, it is still worth exploration and discussion because of the importance of basic skills in school mathematics. Cross-national studies have consistently shown that U.S. students have inadequate knowledge of basic number facts and computational proficiency, especially when compared to Asian students (Cai, 1995; Stigler, Lee, & Stevenson, 1990; U.S. Department of Education, 1997). Would it not be a reasonable expectation for students to develop basic facts and operation skills with numbers and symbols at the same level as higher-order thinking skills in teaching through problem solving?

Final Thoughts

What does research tell us about teaching mathematics through problem solving? This chapter clearly shows that there are some aspects of this approach that have considerable empirical research support, but there also remain some important issues that need additional research. In spite of the absence of research, this approach is receiving increasingly strong support from researchers, educators and teachers. This might be related to the unsatisfactory findings from the national and international assessments of students' mathematical performance, which indicate the urgent need for developing more ambitious learning goals and reforming instructional practices. The traditional ways of teaching, which involve memorizing and reciting facts, rules, and procedures, with an emphasis on the application of well-rehearsed procedures to solve routine problems, are clearly not adequate. Researchers, educators and teachers are eager to reform instructional practices to emphasize the development of students' thinking, understanding, reasoning, and problem solving.

Based on careful analysis of theoretical perspectives and empirical results on teaching mathematics through problem solving, there is a growing consensus among researchers, educators, and teachers that this approach offers considerable promise. Theoretically, this approach makes sense. In teaching through problem solving, learning takes place during the process of problem solving. As students solve problems, they can use any approach they can think of, draw on any piece of knowledge they have learned, and justify their ideas in ways they feel are convincing. The learning environment of teaching through problem solving provides a natural setting for students to present various solutions to their group or class and learn mathematics through social interactions, meaning negotiation, and reaching shared understanding. Such activities help students clarify their ideas and acquire different perspectives of the concept or idea they are learning. Empirically, there are increasing data confirming the promise of teaching through problem solving. However, to actually realize this promise, much more effort, research and development, and refinement of practice must take place. We need to systematically explore issues raised in this chapter as well as issues related to the mechanisms and effectiveness of teaching through problem solving in order to make the promise a reality.

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